e-content for students

B. Sc.(honours) Part 1paper 2

Subject:Mathematics

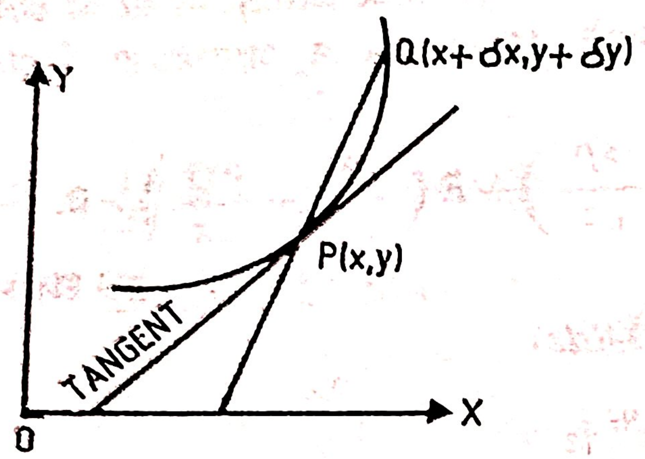
Topic:Tangent & Normal

RRS college mokama

Tangent:Cartesian Eqution

To prove that the equation to the tangent to the curve $y=f(x)$, *at* (x, y) *is* $Y-y=\frac{dy}{dx}(X-x)$ *and that to the curve* $f(x, y)=0$ *is*

$$(X-x)f_x+(Y-y)f_y=0.$$



Let the given point on the curve be $P(x, y)$.

Let a point neighbouring to $P(x, y)$ on the curve be

$$Q(x+\delta x, y+\delta y).$$

So, the equation of PQ is

$$Y-y=\frac{y+\delta y-y}{x+\delta x-x}(X-x)$$

$$=\frac{\delta y}{\delta x}(X-x),$$

where (X, Y) be the current co-ordinates.

Now PQ will become the tangent at P in the limiting position (provided the limit exists) when Q ultimately coincides with P by approaching P along the curve.

So, the equation of the tangent to the curve $y=f(x)$ at $P(x, y)$ will be

$$Y-y=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}(X-x)$$

or

$$\boxed{Y-y=\frac{dy}{dx}(X-x)}$$

From the equation of the curve $f(x, y)=0$,

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}=-\frac{f_x}{f_y}.$$

$\therefore$ The equation to the tangent is

$$Y-y=-\frac{f_x}{f_y}(X-x).$$

i.e.,

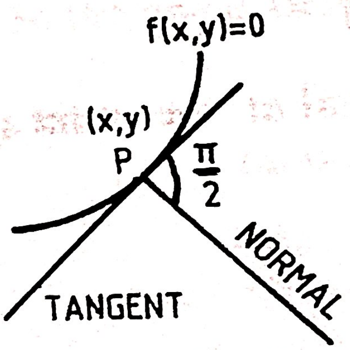
$$\boxed{(X-x)f_x+(Y-y)f_y=0}$$

Normal:Cartesian equation

To prove that the equation of the normal to the curve $f(x, y)=0$ at the point (x, y) is

$$\frac{X-x}{f_x}=\frac{Y-y}{f_y}.$$

and that to the curve $y=f(x)$ is $(X-x)+(Y-y)\frac{dy}{dx}=0$.



We know that the equation of the tangent to the curve $f(x, y)=0$ at $P(x, y)$ is

$$(X-x)f_x+(Y-y)f_y=0.$$

Slope of this tangent at $(x, y)=-\frac{f_x}{f_y}$.

By the normal to the curve at any point we mean the straight line which passes through that point and is at right angles to the tangent at that point.

$\therefore$ Slope of the normal at $(x, y)=\frac{f_y}{f_x}$.

Hence the equation to the normal at the point (x, y) is

$$Y-y=\frac{f_y}{f_x}(X-x).$$

i.e.,

$$\boxed{\frac{X-x}{f_x}=\frac{Y-y}{f_y}}.$$

Q, Find the equation of the tangent to the curve $y=be^{-\frac{x}{a}}$ at the point where the curve crosses the axis of y.

or

Prove that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=be^{-\frac{x}{a}}$ at the point where the curve crosses the axis of y.

Solution. We know that the equation of the y-axis is $x=0$.

The point at which the curve crosses the axis of y is $(0, b)$.

Now the equation of the tangent at $(0, b)$ to the curve is

$$Y-y=(X-x)\frac{dy}{dx} \quad \text{where } y=b,\ x=0.$$

But $\frac{dy}{dx}$ at $y=b$, $x=0$ is $\left[-\frac{b}{a}e^{-\frac{x}{a}}\right]_{\substack{y=b\\x=0}}=-\frac{b}{a}.$

$\therefore$ The equation of the tangent becomes

$$Y-b=(X-0)\left(-\frac{b}{a}\right); \quad \text{or} \quad Y-b=-\frac{b}{a}X$$

or
$$\frac{X}{a}+\frac{Y}{b}=1.$$

If we take (x, y) instead of (X, Y) for the current co-ordinates on the tangent, the equation becomes

$\frac{x}{a}+\frac{y}{b}=1.$ Hence the problem.

Q Prove that the condition that $x\cos\alpha+y\sin\alpha=p$ should touch $x^m y^n=a^{m+n}$ is $p^{m+n}m^m n^n=(m+n)^{m+n}a^{m+n}\sin^n\alpha.\cos^m\alpha.$

Solution. The equation of the tangent at (x, y) is

$$Y-y=\frac{dy}{dx}(X-x).$$

Taking the logarithmic differentiation of $x^m y^n=a^{m+n}$ w.r.t. x,

$$m.\frac{1}{x}+n.\frac{1}{y}\cdot\frac{dy}{dx}=0;\quad \therefore\quad \frac{dy}{dx}=-\frac{m}{n}\cdot\frac{y}{x}.$$

$\therefore$ Equation of the tangent at (x, y) is

$$Y-y=\frac{-m}{n}\cdot\frac{y}{x}(X-x)$$

or $$myX+nxY=mxy+nxy$$

i.e. $$myX + nxY = (m+n)xy. \quad \ldots (1)$$

Now if $X\cos\alpha + Y\sin\alpha = p$ .. (2) be the tangent to the curve at (x, y) then (1) and (2) must be identical.

Comparing, $$\frac{my}{\cos\alpha} = \frac{nx}{\sin\alpha} = \frac{(m+n)xy}{p};$$

$$\therefore \quad \frac{mp}{(m+n)\cos\alpha} = x \text{ and } \frac{np}{(m+n)\sin\alpha} = y.$$

But $x^m y^n = a^{m+n}$; $\therefore$ $$\frac{m^m p^m}{(m+n)^m \cos^m\alpha} \times \frac{n^n p^n}{(m+n)^n \sin^n\alpha} = a^{m+n}$$

or $$p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \sin^n\alpha . \cos^m\alpha.$$